INTEGRABILITY CONDITIONS OF A RESONANT SADDLE IN LIÉNARD-LIKE COMPLEX SYSTEMS

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Abstract. We consider a complex differential system with a resonant saddle that remind the classical Liénard systems in the real plane. For such systems we determine the conditions of analytic integrability of the resonant saddle.

1. Introduction and statement of the main results

The most studied differential equations are the second-order Liénard equations

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

which we rewrite in the equivalent two-dimensional form

$$\dot{x} = y, \qquad \dot{y} = -g(x) - f(x)y. \tag{1}$$

Apart from the fact that system (1) frequently appears in applications of physics, biology and so on, it is studied because many other systems can be transformed into this form, see for instance [7, 12] and references therein. The local question of whether the singular point at the origin of system (1) is a center was studied by Cherkas and later by Christopher and it can be expressed in terms of global conditions of f and g, see [2, 3].

We remark that there are very few results characterizing centers of polynomial or analytic systems of arbitrary degree, see [6]. One of these results characterizes the centers for the Liénard polynomial differential systems of arbitrary degree, see [3]. Another result characterizes the centers of Kukles homogeneous systems, see [8].

Instead of consider that the singular point at the origin is of center-focus type we can consider the case when the singular point is a resonant saddle that is a system of the form

$$\dot{x} = x + X(x, y), \qquad \dot{y} = -y + Y(x, y), \tag{2}$$

where X and Y are analytic functions without constants and linear terms. By using a change of variables of the form $z_1 = x+f(x, y)$ and $z_2 = y+g(x, y)$

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system (2) can be transformed into the Poincaré normal form

$$\dot{z_1} = z_1\Big(1 + \sum_{m=1}^{\infty} a_m (z_1 z_2)^m\Big), \quad \dot{z_2} = -z_2\Big(1 + \sum_{m=1}^{\infty} b_m (z_1 z_2)^m\Big).$$

System (2) is integrable if, and only if, $a_m = b_m$ for all $m \geq 1$ and the number $\alpha_m = a_m - b_m$ is called the m-order saddle constant, see [10] for more details.

Any real differential equation with a singular point of center-focus type can be transformed by complexification into a complex system with a resonant saddle at the origin, see [9]. However the converse in general is not true, see [11].

In this work we consider complex differential systems that have a resonant saddle at the origin and remind the classical Liénard systems in the real plane given by

$$\dot{x} = x, \quad \dot{y} = -y + f(y)x, \tag{3}$$

where f is a formal series in the variable y (that we denote as $f \in \mathbb{R}\{y\}$) and we write it as $f(y) = \sum_{j\geq 1} a_j y^j$. We note that there are very few results characterizing integrable saddles for analytic systems of arbitrary degree. The study of the integrability of resonant saddles became of interest of many authors after the works of Fronville et. al [5] and Żoładek [13] that reveal that new mechanisms of integrability appear in the classification of such systems. In fact, the study of family (3) is used as a toy model because exhibits properties and issues which are important in the classification of more complex systems.

For systems (3) we have the following result.

Theorem 1. *System* (3) *with $f \in \mathbb{R}\{y\}$ has an integrable saddle at the origin if and only if one of the following two conditions holds:*

1) $a_1 = a_2 = 0$;
2) $a_i = 0$ *for* $i \geq 2$.

The conclusion of Theorem 1 is that the center conditions for system (3) are either a finite set of conditions or an infinite set, and as far as we know this is a new phenomena for the center conditions of a resonant saddle.

In the next section we provide the proof of Theorem 1.

2. Proof of Theorem 1

We consider system (3) with $f \in \mathbb{R}\{y\}$. Hence we consider system

$$\dot{x} = x, \qquad \dot{y} = -y + \sum_{j=1}^{\infty} a_j y^j x. \tag{4}$$

The sufficiency of Theorem 1 is proved in the following lemma.

Lemma 2. *System* (4) *with* $a_1 = a_2 = 0$ *or with* $a_i = 0$ *for* $i \geq 2$ *has an analytic first integral defined in a neighborhood of the origin.*

We provide two proofs of very different nature for the existence of an analytic first integral defined in a neighborhood of the origin for system (4) with $a_1 = a_2 = 0$ because both proofs are of independent interest.

Proof. System (4) with $a_1 = a_2 = 0$ takes the form

$$\dot{x} = x, \qquad \dot{y} = -y + \sum_{j=3}^{n} a_j y^j x. \tag{5}$$

In order to prove the existence of an analytic first integral for such system we follow the ideas developed in [1]. We look for a formal first integral expressed of the form $H = \sum_{k=1}^{\infty} h_k(x) y^k$. Imposing that $\dot{H} = 0$ we get,

$$\begin{aligned} 0 &= -h_1(x) + x h_1'(x), \\ 0 &= -2h_2(x) + x h_2'(x), \\ 0 &= a_3 x h_1(x) - 3h_3(x) + x h_3'(x), \\ 0 &= a_4 x h_1(x) + 2a_3 x h_2(x) - 4h_4(x) + x h_4'(x). \end{aligned} \tag{6}$$

and for $k \geq 4$, the functions h_k are determined recursively by the first order differential equation

$$0 = \sum_{i=1}^{k-2} i\, a_{k-i+1}\, x\, h_i(x) - k h_k(x) + x h_k'(x). \tag{7}$$

Solving the first linear differential equation in (7) we obtain

$$h_k(x) = x^k \left(C_k - \int \frac{1}{x^k} \sum_{i=1}^{k-2} i\, a_{k-i+1}\, h_i(x)\, dx \right). \tag{8}$$

Taking into account that the first h_i are polynomial in x of degree i, we assume that the first $k-1$ are polynomial of degree $k-1$, and from (8) we get that h_k is also a polynomial of degree at most k. Hence, system (4) with $a_1 = a_2 = 0$, following [1], has a formal first integral which implies the existence of an analytic one whose power series expansion is $H = xy + \cdots$.

The proof of the integrability of system (5) can also be done in the following more geometrical way. System (5) can be blown down to a node via the transformation $X = xy$. Using this transformation system (5) becomes

$$\dot{X} = y\, X^2\, \frac{f(y)}{y^3}, \qquad \dot{y} = y \left(-1 + X \frac{f(y)}{y^2} \right),$$

and dividing by y, i.e., doing an scaling of the time by $d\tau/dt = y$, we get a non-singular point at the origin. The first integral which exists around the origin by the Flow-box theorem can be pulled back to a first integral of the original system.

System (4) with $a_2 = \ldots = a_n = 0$ has the form

$$\dot{x} = x, \quad \dot{y} = -y + a_1 yx.$$

It has the analytic first integral $H(x, y) = e^{-a_1 x} xy$. $\square$

The necessity condition of Theorem 1 is proved in the following lemma.

Lemma 3. *If system* (3) *with* $f \in \mathbb{R}\{y\}$ *has an integrable saddle at the origin then one of the following two conditions holds:*

1) *either* $a_1 = a_2 = 0$*;*
2) *or* $a_i = 0$ *for* $i \geq 2$*.*

Proof. Assume that system (4) has an integrable saddle at the origin and that $a_1 \neq 0$. In this case we apply the transformation $X = xe^{-a_1 x}$ and the scaling of time $d\tau/dt = 1 - a_1 x$ and system (4) becomes

$$\dot{X} = X, \qquad \dot{y} = \frac{-y + xf(y)}{1 - a_1 x} = -y + \frac{(f(y) - a_1 y)x}{1 - a_1 x}, \tag{9}$$

where $x = x(X)$ is the inverse change. If $f(y) - a_1 y = a_r y^r + \cdots$ with a_r non-zero, then it is straight forward to see that the term in $y^r x^{r-1}$ appears in $\dot{y}$ and it is non-zero. This term will not be removed in the process of transforming system (9) to its normal form (consider the degree of transformation of the form identity plus homogeneous terms which does this). Hence if a_r is non zero, system (9) is normalizable but not integrable (see [4, page 334]) and it can be brought by an analytic change of coordinates $Y = y\phi(x, y)$ with $\phi(0, 0) \neq 0$ into the form

$$\dot{X} = X, \qquad \dot{Y} = -Y(1 + \psi(u)),$$

where $\psi(u) = u^k - au^{2k} + o(u^{3k})$ is a formal function in $u = XY$. Hence the terms $Y^{k+1}X^k$ are the resonant terms of the normal form that cannot be removed.

Thus in order that system (3) with $a_1 \neq 0$ be integrable we must impose that $a_i = 0$ for all $i > 1$ and then we are in the case 2).

Now assume that system (4) has an integrable saddle at the origin and that $a_1 = 0$. It is clear that the resonant term $a_2 y^2 x$ in $\dot{y}$ will not be removed in transforming system (3) into its normal form. Hence a_2 must be zero and we are in case 1). This completes the proof of the lemma. $\square$

Proof of Theorem 1. The proof of Theorem 1 is an immediate consequence of Lemmas 2 and 3. $\square$

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